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ASYMPTOTIC MOTIONS AND THE INVERSION OF THE LAGRANGE-DIRICHLET THEOREM*

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The motions of natural mechanical systems which tend to an equilibrium position as time increases without limit are studied. The degenerate case when several frequencies of small oscillations vanish is explained. An existence theorem is proved for asymptotic trajectories on the assumption that the Maclaurin series for the potential energy has the form $V_2+V_m+V_{m+1}+\ldots$ (V_s is a homogeneous form of degree s) and the function V_2+V_m does not have a local minimum at the equilibrium position. We proved earlier a claim /1, 2/ about the asymptotic motions for the special case when $V_2\equiv 0$. This theorem is used to solve the question of the existence of asymptotic trajectories in the case of simple and unimodal singularities of the potential energy, for which "canonical" normal forms are known. Similar assertions also hold for the equilibrium positions of gradient dynamic systems. The existence of a trajectory, asymptotic to the equilibrium position, naturally implies that this position is unstable in Lyapunov's sense.

1. Introduction. Statement of the results.

Let $x=0$ be a position of equilibrium of a natural mechanical system with potential $V:\mathbf{R}^n\to\mathbf{R}$, $V(0)=0$. In the neighbourhood of the point $x=0$, we can associate the smooth function V with its Maclaurin series $V_2+V_3+\ldots$, which is not necessarily convergent (V_s is a homogeneous form of degree s). We shall study the asymptotic motions, i.e., the non-trivial solutions of the equations of motion $t\mapsto x(t)$, such that $x(t)\to 0$ as $t\to\infty$. Since time is reversible, the function $t\mapsto x(-t)$ is also a motion. Hence the equilibrium is unstable when asymptotic motions are present. If the potential energy has a local minimum at the equilibrium position, there are obviously no asymptotic motions.

The assumption was put forward in /3/ that, if function V is analytic, asymptotic motions exist provided that $x=0$ is not a local minimum point V. In the infinitely differentiable case, the condition that V be analytic can obviously be replaced by the condition that the function $V_2+\ldots+V_k$ should have no minimum for some k. The assumption is not valid without auxiliary constraints of this kind, as is shown by the celebrated Painlevé-Wintner example, see /3/.

The hypothesis of asymptotic motions has been proved in the following cases:

a) $n\leqslant 2$, $x=0$ is an isolated critical point of V;

b) V is a semiquasihomogeneous function in $\mathbf{R}^n$;

c) $V_2=\ldots=V_{m-1}\equiv 0$, while the form V_m has no local minimum at $x=0$.

In cases a) and b), the proof of the existence of asymptotic motions utilizes the following claim, which we shall use below.

Lemma 1 /3/. Let $x=0$ be an isolated critical point of potential V, but not a local minimum point of V. If, in the domain $U_\varepsilon^-=\{x:|x|<\varepsilon,\ V(x)<0\}$, there exists a differentiable vector field v such that

$$(v, V_x')\leqslant 0 \quad \text{in} \quad U_\varepsilon^-$$
$$(v_x'\xi,\xi)\geqslant c\xi^2 \quad \text{for all} \quad \xi\in\mathbf{R}^n \quad \text{and} \quad x\in U_\varepsilon^-,\ c>0$$
$$|v(x)|\to 0 \quad \text{as} \quad x\to 0$$

then the equations of motion have an asymptotic solution.

The case $n=1$ is trivial. In the case $n=2$ the vield v is constructed in /4/ when proving the instability of the equilibrium. In case b) the field v is indicated in /5/. The proof of the existence of asymptotic motion under assumption c) is based on different considerations. Here it is no longer assumed that the critical point $x=0$ is isolated (in the real sense). In the analytic case the asymptotic motions are sought as convergent series, whose form depends essentially on the parity of m

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$$m=2, \quad x(t)=\sum_{i=1}^{\infty} x_i e^{i\lambda t}, \quad \lambda<0, \quad x_i \in \mathbf{R}^n \tag{1.1}$$

$$m=2k+1, \quad x(t)=\sum_{i=1}^{\infty} \frac{x_i}{t^{i\mu}}, \quad \mu=\frac{2}{2k-1}, \quad x_i \in \mathbf{R}^n \tag{1.2}$$

$$m=2k, \quad k \geqslant 2, \quad x(t)=\frac{1}{t^{\mu}} \sum_{i, j=0, j \leqslant \mu i}^{\infty} \frac{x_{ij} (\ln t)^j}{t^{i\mu}} \tag{1.3}$$

$$\mu=\frac{1}{k-1}, \quad x_{ij} \in \mathbf{R}^n$$

The case $m=2$ is Lyapunov's classical result (/6/, Sect.24). The case of odd m is considered in /1/, and the case of even m greater than two, in /2/. In practice the most typical situation is obviously when $m=2$. In certain problems, cases $m>2$ have to be considered. These include e.g., the problem of the stability of a system of charges in an electrostatic field. In this problem every non-trivial form of the Maclaurin expansion (including V_m) is a non-constant harmonic function which takes both positive and negative values. Hence follows, in particular, the strict proof of the well-known Earnshaw theorem on the instability of equilibrium of a system of free charges in a stationary electric field /7/.

If the Lagrange function of the natural mechanical system is infinitely differentiable, but is not analytic in $\mathbf{R}_x^n \times \mathbf{R}_{x'}^n$ (say the function V is non-analytic), then, under the assumptions of case c), the equations of motion again have solutions in the form of series (1.1)-(1.3). However, these series are in general divergent, and in this situation we can speak of the "formal instability" of the equilibrium position. In connection with this remark the interesting problem arises as to whether formal instability implies Lyapunov instability. In the case $m=2$ the answer is well-known to be yest (see e.g., /8/, Chapter 4; the case of equations of dynamics is discussed in detail in /9/). It turns out that the situation is similar for degenerate positions of equilibrium (see Sect.3).

Let us take the more general case when the Maclaurin series of function V has the form $V_2+V_m+V_{m+1}+\ldots$, $m \geqslant 3$. From the point of view of case c) (when $m=2$) it is only worth considering the non-negative form V_2. We know from linear algebra that the set of points of $\mathbf{R}^n$, at which the quadratic form $V_2=0$, forms a k-dimensional plane π, containing the point $x=0$. From the point of view of the theory of small oscillations, precisely k frequencies vanish in this case. It will be assumed henceforth that $k>0$. Otherwise, form V_2 is positive definite, the equilibrium position is stable, and there are no asymptotic motions. Let W_m be the bound of form V_m in the plane π. Clearly, W_m is a homogeneous form of degree m.

Theorem 1. If the function W_m does not have a local minimum at the point $x=0$, then the natural system has motions which are asymptotic at the point $x=0$.

Corollary. Under the assumptions of Theorem 1, the equilibrium $x=0$ is unstable.

If m is odd, the condition tha there is no minimum can be replaced obviously by the condition $W_m \not\equiv 0$. In "typical" cases of degeneracy, form W_3 obviously does not vanish identically. Thus the bifurcation points of general position correspond to unstable equilibrium states (this claim will be refined in Sect.4). In the case when $m=3, k=2$, the instability of the equilibrium was earlier proved in /10/. The case when $k=1$, but m is arbitrary, is considered in /11/ from the point of view of stability; here, in suitable coordinates, the potential V is obviously a semiquasihomogeneous function.

It is well-known that the typical equilibrium positions are non-degenerate critical points of the function V (i.e., form V_2 is non-singular); they are "stable" to small disturbances of the function V. Conversely, degenerate equilibria are destroyed under disturbances of a fairly general kind. The impression may thus be created that the study of degenerate equilibria (their stability, the presence of asymptotic solutions, etc.) leads to nothing significant. This is not the case, however. In practice, the potential usually depends on a certain number of parameters. We know from catastrophe theory that, on disturbance of smooth families of potentials, the degenerate equilibria as a rule do not disappear, but merely change their position slightly, while remaining degenerate (for a disucssion of these questions, see e.g., /12/). By combining Theorem 1 with the classification of simple and unimodal singularities, it can be shown that the hypothesis of asymptotic motions is valid for degenerate equilibria, which reveal themselves inherently in typical families of potentials, dependent on not more than 10 parameters (see Sect.4).

While no attempt is made in the present paper to explain all the formal and analytic aspects of this problem, we to emphasize the importance of analysing degenerate positions of equilibria of natural systems.

2. Formal instability.

Let $V=V_2+V_3+\ldots$. With the quadratic form V_2 can be associated the unique bilinear symmetric form Φ such that $V_2(x)=\Phi(x,x)$. Let π be the zero plane for form Φ, i.e., the set of all vectors $x \in \mathbf{R}^n$ such that $\Phi(x,y)=0$ for all

$y \in \mathbf{R}^n$. If $V_2 \geqslant 0$, then $\pi = \{x: V_2(x) = 0\}$.

Lemma 2. In the neighbourhood of the point $x = 0$ we can introduce coordinates $x_1, \ldots, x_n$, in which

the kinetic energy

$$K = \frac{1}{2}\sum_{i=1}^{n} x_i^{\cdot 2} + \sum_{i,j=1}^{n} a_{ij} x_i^{\cdot} x_j^{\cdot}$$

where a_{ij} are smooth functions of $x_1, \ldots, x_n$, which vanish for $x_1 = \ldots = x_n = 0$;

the potential

$$V = \pm\omega_1^2 x_1^2/2 \pm \ldots \pm \omega_k^2 x_k^2/2 + W(x_{k+1}, \ldots, x_n)$$

where $k = n - \dim \pi$, and W is a smooth function, $W = W_3 + \ldots$.

For the proof, we first have to introduce normal coordinates, then use the lemma on splitting (/13/, Chapter 4). The variables $x_s\ (1 \leqslant s \leqslant k)$ will henceforth often be denoted by $y_1, \ldots, y_k$, and $x_{k+s}\ (1 \leqslant s \leqslant n-k)$ by $z_1, \ldots, z_l\ (k + l = n)$. If $V = V_2 + V_m + \ldots$, and W_m is the bound of form V_m on π, then W_m is the first non-trivial form of the Maclaurin series of the function $z \mapsto W(z)$.

In variables y, z the equations of motion

$$(L_{x^{\cdot}}')^{\cdot} = L_x', \ L = K - V$$

can be written as the system

$$\begin{aligned} y_i^{\cdot\cdot} + (\Gamma_i(x)\, x^{\cdot}, x^{\cdot}) &= \pm\omega_i^2 y_i + \ldots \quad (i \leqslant k) \\ z_j^{\cdot\cdot} + (\Gamma_j(x)\, x^{\cdot}, x^{\cdot}) &= -\partial W_m/\partial z_j + \ldots \quad (j \leqslant l) \end{aligned} \tag{2.1}$$

The unwritten terms in these equations have orders of smallness respectively not less than 2 and m; the symbols $(\Gamma(x)\, x^{\cdot}, x^{\cdot})$ denote quadratic forms in variables $x_s^{\cdot}$ with smooth coefficients, dependent on $x_1, \ldots, x_n$.

Theorem 2. If $W_m: \pi \to \mathbf{R}$ does not have a local minimum at zero, then Eq.(2.1) has solutions in the form of formal series

$$y = \frac{1}{t^{2\mu+2}}\sum_{i=0}^{\infty}\frac{y^{(i)}}{t^{i\mu}}, \quad z = \frac{1}{t^{\mu}}\sum_{i=0}^{\infty}\frac{z^{(i)}}{t^{i\mu}} \tag{2.2}$$
$$\mu = \frac{2}{m-2}, \quad y^{(i)} \in \mathbf{R}^k, \quad z^{(i)} \in \mathbf{R}^l$$

if m is odd, or in the form of formal series

$$y = \frac{1}{t^{2\mu+2}}\sum_{\substack{i,j=0\\ j\leqslant\mu i}}^{\infty}\frac{y^{(ij)}(\ln t)^j}{t^{i\mu}}, \quad z = \frac{1}{t^{\mu}}\sum_{\substack{i,j=0\\ j\leqslant\mu i}}^{\infty}\frac{z^{(ij)}(\ln t)^j}{t^{i\mu}} \tag{2.3}$$
$$\mu = \frac{2}{m-2}, \quad y^{(ij)} \in \mathbf{R}^k, \quad z^{(ij)} \in \mathbf{R}^l$$

if m is even and $m > 2$.

The proof is by induction, increasing with respect to i and decreasing with respect to j (see /1, 2/). The most important step in the proof is to find the coefficients of the series for z. If say the coefficients $z^{(i)}$ up to and including the number s, and the coefficients $y^{(i)}$ up to number $s-1$, are found in series (2.2), then the coefficient $y^{(s)}$ can be found uniquely from the first group of Eqs.(2.1). For, equating coefficients of powers $1/t^{2\mu+2+s\mu}$, we see that the coefficient $y^{(s)}$ can appear only when terms $\pm\omega_i^2 y_i$ are taken into account. The terms of lowest degree $2\mu + 2$ in the series for y appear as a result of terms of the type $A_{\alpha\beta} z_\alpha^{\cdot} z_\beta^{\cdot}\ (A_{\alpha\beta} = \text{const})$ on the left-hand side of the first equation of (2.1). The vectors $z^{(0)}$ and $z^{(00)}$, by means of which all the remaining coefficients are found, are put equal to αe, where $\alpha = \text{const}$, and e is the unit vector on which the form W_m is minimized on the unit sphere $|z| = 1$. The real parameter α is chosen from the condition that the function $\alpha e/t^{\mu}$ satisfies the "simplified" equation of motion (cf. /1/)

$$z^{\cdot\cdot} = -\partial W_m/\partial z, \ z \in \mathbf{R}^l$$

Note that, if $k = 0$, the simplified equation corresponds to the "shortened" system of /14/.

The reason why logarithms appear in series (2.3) with even $m > 2$ can be seen from the example of the system with one degree of freedom

$$x^{\cdot\cdot} + \alpha(x)\, x^{\cdot 2} = 2x^3 + ax^4 + \ldots, \ x \in \mathbf{R} \tag{2.4}$$

The simplified system $x^{\cdot\cdot} = 2x^3$ has the solution $x(t) = 1/t$. We shall seek the asymptotic solution of the complete system (2.4) as the series

$$x = 1/t + x_2/t^2 + x_3/t^3 + \dots \tag{2.5}$$

Substituting this series into (2.4) and collecting terms of order t^{-4}, we obtain the relation $\alpha_0 = a\ (\alpha_0 = \alpha(0))$. Consequently, if $\alpha_0 \neq a$, then Eq.(2.4) has no solutions in the form of power series. Putting $x = 1/t + y$ and assuming that y is small, we obtain from (2.4) a linear inhomogeneous equation, which we can reduce by the substitution $y = ze^{2\tau}$, $t = e^{\tau}$, to an equation with constant coefficients (the prime denotes derivatives with respect to τ)

$$z'' + 3z' - 4z = (a - \alpha_0)e^{-4\tau} \tag{2.6}$$

Among the roots of the characteristic equation is the number -4, so that, with $a \neq \alpha_0$, the solution of (2.6) has to be sought in the form $c\tau e^{-4\tau}$, $c = \text{const}$. Returning to the old variables, we obtain in the expansion of the asymptotic solution the term $(c \ln t)/t^2$. If, however, $\alpha_0 = a$, then Eq.(2.4) has a one-parameter family of asymptotic solutions of type (2.6); the coefficient x_2 is a parameter. Notice that, if system (2.4) admits of the involution $x \mapsto -x$, then $\alpha_0 = a = 0$. This remark can be generalized.

Proposition 1. If $m = 4$ and the Lagrangian $L = K - V$ admits of the involution $x \mapsto -x$, then series (2.3) do not contain logarithms and their coefficients depend on an arbitrary constant.

For even $m > 4$ this claim is false: symmetry of a higher order is required.

We shall seek the solutions of Eqs.(2.1) as the series (2.2). It can be shown (see /1/) that the vector $z^{(i)}$ satisfies the linear equation

$$\left[A - \frac{2(i+1)(2i+m)}{(m-2)^2}E\right]z^{(i)} = a^{(i)}$$

One eigenvalue of the symmetric matrix A is $2m(m-1)/(m-2)^2$, while the remainder are non-positive. The vector $a^{(i)}$ is uniquely defined in terms of the known vectors $z^{(0)}, \dots, z^{(i-1)}$. If $m = 4$, the equation for $z^{(i)}$ can prove to be unsolvable only with $i = 1$. However, $a^{(1)} = 0$, since (since the Lagrangian is even with respect to the variable x) there are no terms of order 4 on the right-hand side of the second equation of (2.1) and $\Gamma_j(0) = 0$. As $z^{(1)}$ we can take any eigenvector of the matrix $(A - 6E)$.

By the defintion of Sect.1, Theorem 2 asserts that the equilibrium $x = 0$ is formally unstable, if the form W_m somewhere takes negative values. Let us emphasize that the coefficient Γ in Eqs.(2.1) need not then necessarily be the same as the Christoffel symbols of the Riemann metric, specified by the kinetic energy. As distinct from series (1.2), (1.3), series (2.2), (2.3) may be divergent (and usually are), even in the analytic case.

The reason for the divergence can easily be seen from the model example:

$$x^{\cdot\cdot} = 12x^2,\quad y^{\cdot\cdot} - x^{\cdot 2} = -y \tag{2.7}$$

In this case $m = 3$ and $W_3 = -4x^3$. System (2.7) has the formal solution

$$x = \frac{1}{2t^2},\quad y = \frac{1}{t^6}\sum_{n=0}^{\infty}\frac{a_{2n}}{t^{2n}},\quad a_{2n} = \frac{(-1)^n(2n+5)!}{120} \tag{2.8}$$

The radius of convergence of the series for y is zero.

Theorem 2 also holds in the infinite-dimensional case, when the co-dimensionality of degeneracy of the quadratic form of the Maclaurin expansion of the potential energy is finite. This assumption holds, for examples, in problems on the oscillations of elastic constructions (/13/, Chapter 13). Admittedly, an elastic rod loaded at the ends retains stability at the first point of bifurcation. In more complicated situations, however, connected with "cracking" of the rod, this is no longer the case: the form $W_3 \not\equiv 0$, and hence the rod loses stability at points of bifurcation. In the case of simple degeneracies, the potential energy is a semi-quasihomogeneous functional, and hence instability of the equilibrium can be deduced from Lemma 1 after suitable generalization.

3. The existence of asymptotic motions. Let us start by analysing the dynamic system of Sect.2, described by Eqs.(2.7). The divergent series for y can be assumed by the following device. Putting $x = 1/(2t^2)$, we obtain a linear differential equation for y: $y^{\cdot\cdot} + y = t^{-6}$. Solving it by the method of varying the constants, we find the solution

$$y(t) = -\sin t\int_t^{\infty}\frac{\cos s}{s^6}\,ds + \cos t\int_t^{\infty}\frac{\sin s}{s^6}\,ds \tag{3.1}$$

which tends to zero as $t \to +\infty$. Performing successively integration by parts, we can obtain from (3.1) the series (2.8). Hence we can naturally regard function (3.1) as the sum (in the generalized sense) of divergent series (2.8). This method of summing divergent series is similar to the familar Borel method /15/. Putting

$$y_N(t)=\frac{1}{t^6}\sum_{n=0}^{N}\frac{a_{2n}}{t^{2n}}$$

we see that $|y(t)-y_N(t)|=O(1/t^{2N})$. Thus the divergent power series (2.8) is the asmyptotic series of function (3.1).

The observations can be extended.

Theorem 3. Under the assumptions of Theorem 2, Eqs.(2.1) have asymptotic solutions, for which series (2.2), (2.3) are the asymptotic expansions.

Let m be odd. The equation of motion $x^{\cdot\cdot}=f(x^{\cdot},x), x\in\mathbf{R}^n$ will be rewritten as the system $x^{\cdot}=v, v^{\cdot}=f(v,x)$, then we make the change of time $t\mapsto\tau$ in accordance with $\tau=1/t^{\alpha}$, $\alpha=1/(m-2)$. Denoting by a prime differentiation with respect to τ, we arrive at the system

$$-\mu\tau^{(\alpha+1)/\alpha}x'=v,\quad -\mu\tau^{(\alpha+1)/\alpha}v'=f(v,x) \tag{3.2}$$

with smooth right-hand side, which has a solution in the form of the formal series $\Sigma x_n\tau^n$. Since $(\alpha+1)/\alpha$ is an integer, we can use the theorem of /16/, which guarantees the existence for system (3.2) of a smooth solution $\tau\mapsto x(\tau$, for which $\Sigma x_n\tau^n$ is its Maclaurin series. If m is even, we can put $\alpha=\mu$. Then, $(\alpha+1)/\alpha=m/2$ is again an integer. In this case, Theorem 3 follows from the theorem of /16/, extended to the case when the formal solution contains powers of $\ln\tau$. The fact that the results of /16/ can be thus extended was pointed out to the author by V.P. Palamodov. Incidentally, only the case $m=4$ will henceforth be considered: with the extra assumption that the Lagrangian is even with respect to the variable x, we can obtain from Proposition 1 and the results of /16/ the existence of an entire family of different asymptotic motions. Theorem 1 follows from Theorems 2 and 3.

In connection with Theorem 3 the interesting problem arises concerning the uniqueness of the solutions of equations of motion with given asymptotic series (2.2), (2.3). The familiar sufficient conditions for uniqueness demand in particular that the coefficient with number n be equal to $O(n!\sigma^n)$ /15, Chapter 8/. These conditions hold for series (2.8). It would appear that we have uniqueness in the analytic case.

4. Simple and unimodal singularities. The classification of degenerate critical points of smooth real functions of several variables is well advanced. In particular, the normal forms of the singular points, which are inherently encountered in smooth families of functions that contain not more than 10 parameters, have been evaluated (all the necessary definitions and results may be found in /12, Chapter 2/).

Theorem 4. If the equilibrium position is a simple or unimodal singular point of the potential and the potential does not have a local minimum at the equilibrium position, then the equations of motion have solutions which are asymptotic to this equilibrium.

In classes of functions of co-dimensionality $c\leqslant 10$, only simple and unimodal singularities are inherently encountered.

The proof of Theorem 4 uses the tables of normal forms of shoots of smooth functions to be found in /12, Sect.17/. In the case of a simple singularity, function W of Lemma 2 reduces to one of the following types: $\pm x^{k+1}\ (k\geqslant 1)$, $x^2y\pm y^{k-1}\ (k\geqslant 4)$, $x^3\pm y^4$, x^3+xy^3, x^3+y^5. In the last four cases $W_3\not\equiv 0$, so that the equilibrium is always unstable. Simple singularities satisfy Theorems 2 and 3. Notice that, in these cases, the potential is a quasihomogeneous function, so that the existence of asymptotic motions can also be deduced from Lemma 1.

The table of unimodal shoots contains 26 distinct types of normal forms. Without quoting them, we merely mention that, in 17 cases, the form $W_3\not\equiv 0$, and in 7 cases W is a semiquasihomogeneous function (so that Lemma 1 is applicable). The two types of singularity $\mathbf{X}_{9+k}$ and $\mathbf{Y}_{r,s}$ deserve special attention. The normal forms of the function W are: $\pm x^4\pm x^2y^2+ay^{4+k}\ (a\neq 0,\ k>0)$ and $\pm x^2y^2\pm x^r+ay^s\ (a\neq 0,\ r,\ s>4)$. In these cases, Theorems 2 and 3 are not always applicable. A simple example is $W=x^4+x^2y^2+y^5$. Let us attempt to use Lemma 1. For this, we first prove.

Proposition 2. Let $V=X+Y$, where X and Y are quasihomogeneous functions of degrees s and r, $0<s<r$, with the same indices of quasihomogeneity $\alpha_1,\ldots,\alpha_n$. We assume that the critical point $x=0$ is isolated and is not a local minimum of V. Then, there is a motion, asymptotic to the point $x=0$, if one of the following conditions also holds:

a) $Y\leqslant 0$ in the domain $\{x\colon V(x)<0\}$,

b) $X\geqslant 0$ in the domain $\{x\colon V(x)<0\}$.

For the proof we take the vector field $v=\Lambda x$, where $\Lambda=\mathrm{diag}(\alpha_1,\ldots,\alpha_n)$. Then, by Euler's formula, $(v,V_x')=sX+rY$. In case a) this is equal to $sV+(r-s)Y$, and in case b), to $-rV+(s-r)X$. It remains to use Lemma 1.

The following is proved by the same method:

Let the analytic potential V be expressible as $V_2+\ldots+V_k+V_{k+1}+\ldots$, where V_s are quasihomogeneous forms of degree s with the same indices of quasihomogeneity. We assume that the point $x=0$ is not a local minimum of V and in the domain $U_\varepsilon^- = \{x: |x| < \varepsilon, V(x) < 0\}$ the forms $V_2 \geqslant 0, \ldots, V_{k-1} \geqslant 0$, while $V_{k+1} \leqslant 0, \ldots$ Then the equilibrium $x=0$ is unstable.

In the case when $\alpha_1 = \ldots = \alpha_n$, this claim is the same as Chetayev's familiar result /17/.

For clarity, take the singularity $\mathbf{X}_{9+k}$. If the term x^2y^2 appears with the plus sign, we put $X = z_1^2 + \ldots + z_k^2 + x^2y^2$, $Y = \pm x^4 + ay^{4-k}$ The functions X and Y are quasihomogeneous of degree $2(4+k)+8$ and $4(4+k)$ with indices of quasihomogeneity $4+k$ and 4 with respect to the variables x, y and with indices $8+k$ with respect to z_s. Since $X \geqslant 0$, we can apply Proposition 2. We now put $X = z_1^2 + \ldots + z_k^2 \pm x^4 - x^2y^2, Y = ay^{4+k}$. Clearly, the form $Y \leqslant 0$ in one of the connected components of the domsin $\{V < 0\}$, if k is odd, or k is even and $a < 0$. In these cases we can again use Proposition 2. The case $W = \pm x^4 - x^2y^2 + ay^{4+2k}$, $a > 0$, $k > 0$. remains unconsidered. However, the form $W_4 = -x^2y^2 \pm x^4$ has not a local minimum at the point $x = y = 0$, so that Theorems 2 and 3 are applicable. Moreover, if the coeffidients of the quadratic form K are even functions, we can use Proposition 1. The singularity $\mathbf{Y}_{r,s}$ can be treated similarly.

5. Thom's problem.

A system of the type

$$\sum_{j=1}^{n} g_{ij} x_j^{\cdot} = -\frac{\partial V}{\partial x_i} \tag{5.1}$$

is called a gradient dynamic system. Here, $g_{ij} = g_{ji}$ are coefficients of a metric tensor, smoothly dependent on $x_1, \ldots, x_n$, and $V\colon \mathbf{R}^n \to \mathbf{R}$ is a smooth function which will henceforth be called the potential. We again assume that $dV(0) = 0$ and $V(0) = 0$. Gradient systems seem to have first been considered by Lyapunov in connection with the analytis of the stability of equilibrium positions (/4/, Sect.16). They were then studied by Smale in the theory of structural stability /18/, and by Thom and his successors in catastrophe theory /19/. It happens that we can apply to them the arguments used above for natural mechanical systems.

Proposition 3. Let V be an analytic function which does not have a local minimum at the point $x=0$. Then, the equilibrium $x=0$ is unstable. Under the extra assumption that the critical point $x=0$ is isolated (in the real sense), system (5.1) has a solution $t \mapsto x(t)$ such that $x(t) \to 0$ and $t \to -\infty$.

For the proof, we use the equation

$$V^{\cdot} = \sum_{i,j=1}^{n} g^{ij} \frac{\partial V}{\partial x_i} \frac{\partial V}{\partial x_j}$$

where $\|g^{ij}\|$ is a positive definite matrix inverse to $\|g_{ij}\|$. Since the function V is analytic, there are no critical points /20/ in a small neighbourhood of zero in the domain $\{V(x) < 0\}$. Hence V is a Chetayev function and the equilibrium $x=0$ is unstable. A more significant claim about the existence of an asymptotic solution departing from the point $x=0$ follows from Krasovskii's theorem and the instability (/21, Sect.13/). It is not clear whether asymptotic solutions always exist when the critical point $x=0$ is not isolated. Notice that, if V is analytic at the point $x=0$, which is a local minimum of V, then the equilibrium $x=0$ is asymptotically stable (see /4, Para.16/).

Thom put forward the hypothesis of the existence (in the conditions of Propostion 3) of asymptotically departing trajectories with the limiting tangential direction

$$\lim_{t \to -\infty} \frac{x^{\cdot}(t)}{|x^{\cdot}(t)|} \tag{5.2}$$

As far as the present author knows, this problem has still not been fully solved. For non-degenerate critical points, exhaustive information about the asymptotic solutions can be deduced from the results of /9/.

Lemma 3. In certain coordinates $x = (y_1, \ldots, y_k, z_1, \ldots, z_l)$, $k + l = n$, the coefficients $g_{ij} = \delta_{ij} + a_{ij}(x)$, $a_{ij}(0) = 0$, while $V = \pm \omega_1^2 y_1^2/2 \pm \ldots \pm \omega_k^2 y_k^2/2 + W(z_1, \ldots, z_l)$, where $W = W_3 + \ldots$

This claim is just the same as Lemma 2.

Theorem 5. Let $W = W_m + W_{m+1} + \ldots$ and let from W_m have no local minimum at zero. Then, asymptotic solutions of Eq.(5.1) exist, whose asymptotic expansions are

$$y = \frac{1}{t^{2\mu+1}} \sum_{i,j=0,\, j \leqslant \mu i} \frac{y^{(ij)} (\ln t)^j}{t^{i\mu}} \tag{5.3}$$

$$z = \frac{1}{t^{\mu}} \sum_{i,j=0,\, j \leqslant \mu i} \frac{z^{(ij)} (\ln t)^j}{t^{i\mu}}, \quad \mu = \frac{1}{m-2}$$

The proof is the same as in Theorems 2 and 3. The coefficients of the first terms of series (5.3) give precisely the limiting position of tangent (5.2). The logarithms of t in series (5.3) appear independently of the parity of m. In special cases there may be no logarithms; the coefficients in (5.3) then contain "moduli", i.e., arbitrary real constants. If $k=0$, then series (5.3) are convergent in the analytic case (see /2/).

From Theorem 5 we can obtain

Corollary. Thom's hypothesis is certainly valid for degenerate critical points, which inherently appear in families of functions which depend on not more than 9 parameters.

For singularities of the type $\mathbf{X}_{9+k}$ and $\mathbf{Y}_{r,s}$, Theorem 5 is not in general applicable. However, the co-dimensionality of these classes of functions is not less than 10.

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